

# RILEY'S CONJECTURE ON $SL(2, \mathbb{R})$ REPRESENTATIONS OF 2-BRIDGE KNOTS

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**1. Introduction.** In [R1] and [R2] Riley investigated representations of 2-bridge knot groups in  $SL(2, F)$  for various fields  $F$ . In particular, he considered non-abelian representations in which the meridians go to parabolic elements, calling these *parabolic* representations. He showed that, for a given 2-bridge knot  $K$ , such representations correspond to the roots of a certain polynomial  $\lambda_K(x) \in \mathbb{Z}[x]$ , the *Riley polynomial*; see [R1, Theorem 2]. Thus the real roots of  $\lambda_K(x)$  give parabolic  $SL(2, \mathbb{R})$  representations. In [R2], Riley states “Some of our computer calculations made in 1972-73 ... suggested that the number of real roots of  $[\lambda_K(x)]$  is not less than  $|\sigma|/2$ .” Here  $\sigma = \sigma(K)$  is the signature of  $K$ . Following [Tr2], we will refer to this as the

**Riley Conjecture.** *The number of real roots of the Riley polynomial of a 2-bridge knot  $K$  is at least  $|\sigma(K)|/2$ .*

Note that since  $\lambda_K(x)$  has no multiple roots [R1, Theorem 3], the statement is unambiguous.

Our main result is

**Theorem 1.1.** *The Riley Conjecture is true.*

For double twist knots, the Riley Conjecture was proved by Tran [Tr2].

One of our interests in the Riley Conjecture is its connection with the question of when the  $n$ -fold cyclic branched cover  $\Sigma_n(K)$  of a knot  $K$  has left-orderable fundamental group. More precisely, as pointed out in [Tr2], Hu’s argument in [H] shows that Theorem 1.1 has the following corollary.

**Corollary 1.2.** *Let  $K$  be a 2-bridge knot with  $\sigma(K) \neq 0$ . Then  $\Sigma_n(K)$  has left-orderable fundamental group for  $n$  sufficiently large.*

By contrast, there are 2-bridge knots  $K$  such that  $\Sigma_n(K)$  has non-left-orderable fundamental group for all  $n$ , by [Te, Proof of Theorem 2] and [BGW, Theorem 4].

For any knot  $K$ , the determinant and signature are related by the following congruence [M, Theorem 5.6]

$$\det(K) \equiv (-1)^{\sigma(K)/2} \pmod{4}$$

If  $K$  is the 2-bridge knot corresponding to  $p/q \in \mathbb{Q}$ ,  $p > 0$ , then  $\det(K) = p$ . Hence if  $p \equiv -1 \pmod{4}$  then  $\sigma(K) \equiv 2 \pmod{4}$ , and Corollary 1.2 applies. In this case the conclusion of Corollary 1.2 was proved by Hu [H].

For other results on the left-orderability of the fundamental groups of cyclic branched covers of knots see [GL] and [Tr1].

**Question 1.3.** *Does Corollary 1.2 hold without the assumption that  $K$  is 2-bridge?*

The proof of Theorem 1.1 uses a variant of the classical theorem of Sturm on the number of real roots of a polynomial with real coefficients. This is treated in Section 2. In Section 3 we prove the Riley Conjecture, and in Section 4 we discuss Corollary 1.2.

*Acknowledgements.* I would like to thank Steve Boyer and Anh Tran for helpful conversations. This research was partially supported by NSF Grant DMS-1309021.

**2. Sturm's theorem.** Sturm's theorem gives a way of determining the number of real roots of a polynomial with real coefficients; for a nice discussion of this, including some history, see [GR]. The method depends on constructing a sequence of polynomials  $f_0, f_1, \dots, f_n = f$  with certain properties (we find it convenient to reverse the usual numbering convention). In Theorem 2.1 we prove a version of Sturm's theorem where the key properties of  $f_0, f_1, \dots, f_{n-1}$  are as in the classical setting, but the hypothesis on the relation between  $f_n$  and  $f_{n-1}$  is weakened. The conclusion is then an inequality rather than an equality.

Let  $\alpha = (\alpha_k) = (\alpha_0, \alpha_1, \dots, \alpha_n)$  be a sequence of non-zero real numbers. Define the *variation*  $\text{var}(\alpha)$  of  $\alpha$  to be the number of changes in the corresponding sequence of signs  $\text{sign}(\alpha) = (\text{sign}(\alpha_k))$ , i.e.

$$\text{var}(\alpha) = \#\{k : \alpha_k \alpha_{k+1} < 0, 0 \leq k < n\}$$

Let  $\mathbf{f} = (f_0, f_1, \dots, f_n)$  be a sequence of polynomials in  $\mathbb{R}[X]$ . If  $x \in \mathbb{R}$ , set  $\mathbf{f}(x) = (f_0(x), f_1(x), \dots, f_n(x)) \in \mathbb{R}^{n+1}$ .

Let  $Z_k = \{\text{real roots of } f_k\} \subset \mathbb{R}$ ,  $0 \leq k \leq n$ , and let  $Z = \bigcup_{k=0}^n Z_k$ . Choose  $x_+$  and  $x_- \in \mathbb{R}$  such that  $Z \subset (x_-, x_+)$ . Then  $\text{sign}(\mathbf{f}(x_+))$  is independent of the choice of  $x_+$ , so we denote it by  $\text{sign}(\mathbf{f}(\infty))$ , and write  $\text{var}(\mathbf{f}(\infty)) = \text{var}(\mathbf{f}(x_+))$ . Similarly, we write  $\text{sign}(\mathbf{f}(-\infty)) = \text{sign}(\mathbf{f}(x_-))$  and  $\text{var}(\mathbf{f}(-\infty)) = \text{var}(\mathbf{f}(x_-))$ .

**Theorem 2.1.** *Let  $\mathbf{f} = (f_0, f_1, \dots, f_n)$  be a sequence of polynomials in  $\mathbb{R}(X)$  such that*

- (1)  *$f_0$  is constant and non-zero, and*
- (2) *if  $f_k(x_0) = 0$  for some  $0 < k < n$  and  $x_0 \in \mathbb{R}$ , then  $f_{k-1}(x_0)f_{k+1}(x_0) < 0$ .*

*Then  $f_n$  has at least  $|\text{var}(\mathbf{f}(-\infty)) - \text{var}(\mathbf{f}(\infty))|$  distinct real roots.*

*Proof.* The theorem is trivially true when  $n = 0$  so we assume  $n \geq 1$ .

Define  $v : \mathbb{R} \setminus Z \rightarrow \{0, 1, \dots, n\}$  by  $v(x) = \text{var}(\mathbf{f}(x))$ . Note that  $v$  is constant on each component of  $\mathbb{R} \setminus Z$ .

Suppose  $x_0 \in Z$ , so  $x_0 \in Z_k$  for some  $k$  (not necessarily unique) with  $1 \leq k \leq n$ .

If  $k < n$  then by condition (2) there exists  $\delta > 0$  such that  $f_{k-1}(x)f_{k+1}(x) < 0$  for all  $x \in (x_0 - \delta, x_0 + \delta)$ . Hence, as  $x$  passes through  $x_0$  the signs of  $(f_{k-1}(x), f_k(x), f_{k+1}(x))$  change as  $(\pm, \epsilon, \mp) \rightarrow (\pm, \epsilon', \mp)$ , where  $\epsilon, \epsilon' \in \{+, -\}$ . This contributes 0 to the change in  $v(x)$ .

Suppose  $k = n$ . Note that  $f_{n-1}(x_0) \neq 0$ , by (1) if  $n = 1$  and by (2) if  $n > 1$ . Hence, as  $x$  passes through  $x_0$  the signs of  $(f_{n-1}(x), f_n(x))$  change as  $(\pm, \epsilon) \rightarrow (\pm, \epsilon')$ . Thus the corresponding change in  $v(x)$  is 0 or  $\pm 1$ .

Therefore  $|\text{var}(\mathbf{f}(-\infty)) - \text{var}(\mathbf{f}(\infty))|$  is at most the number of distinct real roots of  $f_n$ .  $\square$

**3. The Riley Conjecture.** Let  $K$  be the 2-bridge knot corresponding to  $p/q \in \mathbb{Q}$ , where  $p$  and  $q$  are coprime, and  $p$  is odd and  $> 1$ . Let  $n = (p-1)/2$ . Then (see [R1, Proposition 1]) there exist  $\epsilon_i, \eta_i \in \{\pm 1\}$ , with  $\epsilon_i = \eta_{n+1-i}$ ,  $1 \leq i \leq n$ , such that  $\pi(K) = \pi_1(S^3 \setminus K)$  has presentation

$$\langle a, b : wa = bw \rangle,$$

where  $a$  and  $b$  are meridians and  $w = \prod_{i=1}^n a^{\epsilon_i} b^{\eta_i}$ .

Also,  $\sigma(K) = \sum_{i=1}^n (\epsilon_i + \eta_i)$  [S]. Hence  $\sum_{i=1}^n \epsilon_i = \sigma(K)/2$ .

Riley considers parabolic representations of  $\pi(K)$  into  $SL(2, \mathbb{C})$ , where

$$a \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = A, \text{ and}$$

$$b \rightarrow \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = X.$$

Let  $W_k = \prod_{i=1}^k A^{\epsilon_i} X^{\eta_i}$ ,  $1 \leq k \leq n$ , and set  $W_0 = I$ .

Write  $W_k = \begin{pmatrix} a_k & b_k \\ * & * \end{pmatrix}$ ,  $a_k, b_k \in \mathbb{Z}[x]$ ,  $0 \leq k \leq n$ .

The *Riley polynomial* of  $K$  is defined to be  $\lambda_K = a_n$ . Riley showed [R1, Theorem 2] that the above assignment of  $a$  and  $b$  defines a homomorphism from  $\pi(K)$  to  $SL(2, \mathbb{C})$  if and only if  $\lambda_K(x) = 0$ . Thus the real roots of  $\lambda_K$  give parabolic representations of  $\pi(K)$  into  $SL(2, \mathbb{R})$ .

Let  $\delta_i = \epsilon_i \eta_i$ ,  $1 \leq i \leq n$ . Then

$$A^{\epsilon_i} X^{\eta_i} = \begin{pmatrix} 1 + \delta_i x & \epsilon_i \\ \eta_i & 1 \end{pmatrix},$$

giving the recurrence equations, for  $1 \leq k \leq n$ ,

$$(3.1) \quad a_k = (1 + \delta_k x) a_{k-1} + (\eta_k x) b_{k-1}$$

$$(3.2) \quad b_k = \epsilon_k a_{k-1} + b_{k-1}$$

It follows from (3.1) and (3.2) by induction on  $k$  that  $a_k$  has degree  $k$ , with leading coefficient  $\prod_{i=1}^k \delta_i$ , and

$$(3.3) \quad a_k(0) = 1$$

Also, since  $\det W_k = 1$ , we have that for all  $x \in \mathbb{C}$ ,

$$(3.4) \quad a_k(x) \text{ and } b_k(x) \text{ are not both zero, } 0 \leq k \leq n$$

**Lemma 3.1.** *If  $0 < k < n$  and  $a_k(x_0) = 0$ ,  $x_0 \in \mathbb{R}$ , then  $a_{k-1}(x_0)$  and  $a_{k+1}(x_0)$  are non-zero and  $\text{sign}(a_{k-1}(x_0))\text{sign}(a_{k+1}(x_0)) = -\eta_k \eta_{k+1}$ .*

*Proof.* Suppose  $a_k(x_0) = 0$ . Then (3.1) gives

$$(3.5) \quad (1 + \delta_k x_0) a_{k-1}(x_0) + (\eta_k x_0) b_{k-1}(x_0) = 0$$

while from (3.2) we get

$$(3.6) \quad b_k(x_0) = \epsilon_k a_{k-1}(x_0) + b_{k-1}(x_0)$$

Multiplying both sides of (3.6) by  $\eta_k x_0$  and using (3.5) gives

$$(3.7) \quad a_{k-1}(x_0) = -(\eta_k x_0) b_k(x_0)$$

Replacing  $k$  by  $k + 1$  in (3.1) we obtain

$$(3.8) \quad a_{k+1}(x_0) = (\eta_{k+1} x_0) b_k(x_0)$$

By (3.3)  $x_0 \neq 0$ , and by (3.4)  $b_k(x_0) \neq 0$ . The result now follows from (3.7) and (3.8).  $\square$

*Proof of Theorem 1.1.* Define  $f_k = (\prod_{i=1}^k \eta_i) a_k$ ,  $0 \leq k \leq n$ . Then  $f_0$  is the constant polynomial 1, and Lemma 3.1 implies that if  $f_k(x_0) = 0$  for some  $0 < k < n$  then  $f_{k-1}(x_0) f_{k+1}(x_0) < 0$ . Thus  $\mathbf{f} = (f_k)$  satisfies the hypotheses of Theorem 2.1.

The coefficient of  $x^k$  in  $f_k$  is  $(\prod_{i=1}^k \eta_i)(\prod_{i=1}^k \delta_i) = \prod_{i=1}^k \epsilon_i = \mu_k$ , say.

Then  $\text{sign}(\mathbf{f}(\infty)) = (\mu_k)$ , and  $\text{sign}(\mathbf{f}(-\infty)) = ((-1)^k \mu_k)$ . Since  $\mu_k = \epsilon_k \mu_{k-1}$ , we have

$$\text{var}(\mathbf{f}(\infty)) = \#\{k : 1 \leq k \leq n, \epsilon_k = -1\}$$

and

$$\text{var}(\mathbf{f}(-\infty)) = \#\{k : 1 \leq k \leq n, \epsilon_k = +1\}$$

Therefore by Theorem 2.1 the number of real roots of  $\lambda_K = a_n = \pm f_n$  is at least

$$|\text{var}(\mathbf{f}(-\infty)) - \text{var}(\mathbf{f}(\infty))| = \left| \sum_{k=1}^n \epsilon_k \right| = |\sigma(K)|/2.$$

$\square$

*Remark.* The inequality in the Riley Conjecture can be strict. For example, the knot  $10_{32}$ , which is the 2-bridge knot corresponding to the rational number  $69/29$ , has  $\sigma(10_{32}) = 0$ . On the other hand, by [KT1] and [KT2] (see also [ORS]), there is a meridian-preserving epimorphism from  $\pi(10_{32})$  to  $\pi(3_1)$ , the group of the trefoil. Since  $\pi(3_1)$  has a real parabolic representation [R1], so does  $10_{32}$ .

This also shows that the converse of Corollary 1.2 is not true. In fact, by [GL, Theorem 1.2 and Lemma 9.1],  $\Sigma_n(10_{32})$  has left-orderable fundamental group for  $n \geq 6$ .

**4. Cyclic branched covers.** In this section we indicate how the argument in [H] gives Corollary 1.2.

In [R2] Riley considers arbitrary non-abelian  $SL(2, \mathbb{C})$  representations of  $\pi(K)$ ,  $K$  a 2-bridge knot. Up to conjugation, we may assume that

$$a \rightarrow \begin{pmatrix} t & 1 \\ 0 & t^{-1} \end{pmatrix}, \text{ and} \\ b \rightarrow \begin{pmatrix} t & 0 \\ x & t^{-1} \end{pmatrix}.$$

Riley shows that this defines a representation if and only if  $\phi(t, x) = 0$  for a certain polynomial  $\phi \in \mathbb{Z}[t^{\pm 1}, x]$ . He notes that  $\phi(t, x) = \phi(t^{-1}, x)$  [R2, Proposition 1], and therefore  $\phi(t, x) = \psi(s, x)$ , where  $s = t + t^{-1}$ , for some  $\psi \in \mathbb{Z}[s, x]$ . Then  $\psi(2, x)$  is the Riley polynomial  $\lambda_K(x)$ .

Suppose  $\lambda_K$  has a real root  $x_0$ . Since  $\lambda_K$  has no repeated factors [R1, Theorem 3],  $\frac{\partial \psi}{\partial x}|_{(s=2, x=x_0)}$  is non-zero. It follows that there exists  $\delta > 0$  and a continuous function  $\gamma : (2 - \delta, 2 + \delta) \rightarrow \mathbb{R}$ , with  $\gamma(2) = x_0$ , such that  $\psi(s, \gamma(s)) = 0$  for all  $s \in (2 - \delta, 2 + \delta)$ . In particular, for all  $s \in (2 - \delta, 2)$  there is a non-abelian representation  $\rho_s : \pi(K) \rightarrow SL(2, \mathbb{R})$  such that  $\rho_s(a)$  has trace  $s$ . Conjugating  $\rho_s$  we may assume that

$$\rho_s(a) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

where  $s = 2 \cos \theta$ .

For  $n$  sufficiently large,  $s_n = 2 \cos(2\pi/n) \in (2 - \delta, 2)$ . Then  $\rho_{s_n}(a)$  has order  $n$ . It follows from [H, Theorem 3.1] (see also [BGW, Theorem 6]) that  $\pi_1(\Sigma_n(K))$  is left-orderable.

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